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A Multi-Player Potential Game Approach for Sensor Network Localization with Noisy Measurements

Gehui Xu[1], Guanpu Chen[2], Baris Fidan[3], Yiguang Hong[4], Hongsheng Qi[5], Thomas Parisini[6], and Karl H. Johansson[2]

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***Abstract*— Sensor network localization (SNL) is a challenging problem due to its inherent non-convexity and the effects of noise in inter-node ranging measurements and anchor node position. We formulate a non-convex SNL problem as a multi-player non-convex potential game and investigate the existence and uniqueness of a Nash equilibrium (NE) in both the ideal setting without measurement noise and the practical setting with measurement noise. We first show that the NE exists and is unique in the noiseless case, and corresponds to the precise network localization. Then, we study the SNL for the case with errors affecting the anchor node position and the inter-node distance measurements. Specifically, we establish that in case these errors are sufficiently small, the NE exists and is unique. It is shown that the NE is an approximate solution to the SNL problem, and that the position errors can be quantified accordingly. Based on these findings, we apply the results to case studies involving only inter-node distance measurement errors and only anchor position information inaccuracies.**

I. INTRODUCTION

Accurate information regarding the location of nodes within wireless sensor networks (WSNs) is essential in diverse applications, such as target tracking and detection [1], environment monitoring [2], area exploration [3], data collection, as well as cooperative robot tasks [4]. A common approach for sensor localization involves utilizing (noisy) ranging information obtained through signal transmission techniques, such as time of arrival, time-difference of arrival, and strength of received radio frequency signals [5]. Also, there are anchor nodes with known global positions [6]. Then a sensor network localization (SNL) problem is defined as follows: Given the positions of the anchor nodes of the WSN and the measurable information among each non-anchor node and its neighbors, find the positions of the rest of non-anchor nodes.

In localization tasks, sensor nodes use distance measurements and computational abilities to estimate their positions through interactions, enabling the SNL problem to be formulated with game theory [7]–[9]. The Nash equilibrium (NE) is a key solution concept characterizing a profile of stable strategies in which rational players would not choose to deviate from their own strategies [10], [11]. Specifically, potential games are well-suited to describe individual preferences and network interactions in SNL (see [7], [12]). The potential game paradigm guarantees an alignment between the individual sensor profits and the network objective by exploiting a unified potential function. Then, it is possible to find the NE corresponding to the global optimum for the whole WSN rather than a local approximation.

It is worth noting, though, that the effects of the noises in inter-node ranging measurements and the inaccuracies in anchor node position information are an intrinsic challenge of SNL problems, which, unfortunately, cannot be readily avoided by potential games or other modeling approaches. The inter-node distance measurements between sensor nodes are often subject to inevitable uncertainties like transmission interference or time delay, leading to measurement errors [13]. Besides, many existing studies assume precise anchor positions to estimate the location of the rest of the sensor nodes. However, in many scenarios, anchor positions may not be accurately known. This is often attributed to the reliance on the Global Positioning System (GPS) or other positioning systems for determining anchor positions, which may cause estimation errors [14].

A fundamental problem in such noisy SNL problems is whether a given sensor network can be uniquely localized. The uniqueness of localizability is usually revealed using graph rigidity theory [15]. When the measurement information is exact, the generically global rigidity of the grounded graph guarantees the uniqueness of NE, corresponding to the precise localization of the network. However, in the presence of noise, there may exist multiple sets of non-congruent

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[1]Gehui Xu is with the Department of Electrical and Electronic Engineering, Imperial College London, London SW7 2AZ, UK. g.xu@imperial.ac.uk

[2]Guanpu Chen and Karl H. Johansson are with the Division of Decision and Control Systems, School of Electrical Engineering and Computer Science, KTH Royal Institute of Technology, 100 44, Stockholm, Sweden. guanpu@kth.se, and kallej@kth.se

[3]Baris Fidan is with the Department of Mechanical and Mechatronics Engineering, University of Waterloo, Waterloo, ON N2L 3G1, Canada. fidan@uwaterloo.ca

[4]Yiguang Hong is with Department of Control Science and Engineering, Tongji University, Shanghai 201804, China, and is also with Shanghai Research Institute for Intelligent Autonomous Systems, Shanghai 201210 China. yghong@iss.ac.cn

[5]Hongsheng Qi is with Key Laboratory of Systems and Control, Academy of Mathematics and Systems Science, Beijing, China, and also with School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing, China. qihongsh@amss.ac.cn

[6]Thomas Parisini is with the Department of Electrical and Electronic Engineering, Imperial College London, London SW7 2AZ, UK, with the Department of Electronic Systems, Aalborg University, Denmark, and with the Department of Engineering and Architecture, University of Trieste, Italy. t.parisini@imperial.ac.uk

sensor localizations that satisfy the provided inaccurate measurements. The presence of noise may perturb the rigidity of the graph, resulting in flip ambiguities or even no solution to the problem. Such a localization problem with measurement errors has been considered in some existing works. Lui et al. [14] studied an uncertain SNL problem with Gaussian distributed disturbances and used the semi-definite programming relaxation technique to solve the formulated maximum likelihood estimation problem. Then, Naddafzadeh-Shirazi et al. [16] considered a similar formulation and employed second-order cone programming to seek a robust solution with complexity reduction. On the other hand, Anderson et al. [6] investigated an SNL problem with errors in inter-node distance measurements and transformed it into a minimization problem to reveal the relation between errors in positions and errors in distance measurements. However, [14] and [16] do not formally study how noise affects the deviation of sensor positions from the true values, while [6] do not consider the uncertainty in anchor node positions.

The main objective of this paper is to study the solution to the noisy SNL problem. We focus on the case when the inter-node distance measurements and anchor position information are subject to some errors, presumably on account of the measurement process. Specifically, we formulate the non-convex SNL problem as a potential game and investigate the existence and uniqueness of NE in both the ideal setting with accurate anchor node location information and accurate inter-node distance measurements and the practical setting with anchor location inaccuracies and distance measurement noise. We first show that the NE exists and is unique in the noiseless case, corresponding to the precise network localization. Then we study SNL for the case with errors in anchor node position and inter-node distance measurements. We establish that if these errors are sufficiently small, the NE exists and is unique. It is shown that the NE is an approximate solution to the SNL problem, and that the position errors can be quantified accordingly. Then, we apply the results to case studies involving only inter-node distance measurement errors and only anchor position information inaccuracies.

II. Problem formulation

In this section, we first introduce the noisy inter-node range measurement based SNL problem of interest and then formulate it as a potential game.

A. Noiseless SNL problem

Consider a static sensor network in $\mathbb{R}^2$ composed of M anchor nodes whose positions are known and N non-anchor sensor nodes whose positions are unknown ($M < N$). Let a graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ represent the sensing relationships between sensors, where $\mathcal{N}$ is the sensor node set and $\mathcal{E} \subseteq \mathcal{N} \times \mathcal{N}$ is the set of edges that represent the sensor node pairs whose range measurements are available. Specifically, $\mathcal{N} = \mathcal{N}_s \cup \mathcal{N}_a = \{i\}_{i=1}^{\bar{N}}$, where $\bar{N} = N + M$, $\mathcal{N}_s = \{1, \ldots, N\}$ and $\mathcal{N}_a = \{N+1, \ldots, N+M\}$ correspond to the sets of non-anchor nodes and anchor nodes, respectively. Let $x_i^\star \in \mathbb{R}^2$ denote the actual position of sensor node $i \in \mathcal{N}$ with $\boldsymbol{x}^\star \triangleq \text{col}\{x_i\}_{i\in\mathcal{N}}$. For a pair of sensor nodes i and j, their Euclidean distance is denoted by $d_{ij}^\star$. Each sensor has the capability of sensing range measurements from other sensors within a fixed range R_s, and $\mathcal{E} = \mathcal{E}_{ss} \cup \mathcal{E}_{as} \cup \mathcal{E}_{aa}$ with $\mathcal{E}_{ss} = \{(i,j) \in \mathcal{N}_s \times \mathcal{N}_s : \|x_i^\star - x_j^\star\| \le R_s, i \ne j\}$ denoting the edge set between non-anchor nodes, $\mathcal{E}_{as} = \{(i,l) \in \mathcal{N}_s \times \mathcal{N}_a : \|x_i^\star - x_l^\star\| \le R_s\}$ denoting the edge set between anchor nodes and non-anchor nodes, $\mathcal{E}_{aa} = \{(l,m) \in \mathcal{N}_a \times \mathcal{N}_a, l \ne m\}$ denoting the edge set between anchor nodes. The range-based SNL task in the noiseless case is to determine the accurate positions of all non-anchor sensor nodes $i \in \mathcal{N}_s$ when all anchor node positions $x_l^\star$, $l \in \mathcal{N}_a$ and measurements $d_{ij}^\star$, $d_{il}^\star$ are given:

$$\begin{aligned} \text{find} \quad & x_1, \ldots, x_N \in \mathbb{R}^2 \\ \text{s.t.} \quad & \|x_i - x_j\|^2 = d_{ij}^{\star 2}, \forall (i,j) \in \mathcal{E}_{ss}, \\ & \|x_i - x_l^\star\|^2 = d_{il}^{\star 2}, \forall (i,l) \in \mathcal{E}_{as}. \end{aligned} \tag{1}$$

Denote $\boldsymbol{x}^\dagger = \text{col}\{x_1^\dagger, \ldots, x_N^\dagger\} \in \mathbb{R}^{2N}$ as the solution to noiseless SNL problem (1).

B. Noisy SNL problem

In practical scenarios, the anchor positions are usually obtained through GPS or other positioning systems, thereby causing position information inaccuracies. For $l \in \mathcal{N}_a$, let anchor node l's position be measured as

$$x_l = x_l^\star + \epsilon_l, \quad l \in \mathcal{N}_a,$$

where $x_l^\star$ is the actual position of l and $\epsilon_l \in \mathbb{R}^2$ represents the position information inaccuracy.

Consider the errors in the squares of inter-node distance measurements, similar to [6]. For each non-anchor node pair $(i,j) \in \mathcal{E}_{ss}$, the square of the measured distance between them is denoted by

$$d_{ij}^2 = d_{ij}^{\star 2} + \mu_{ij},$$

where $\mu_{ij} \in \mathbb{R}$ represents the measurement error. For each anchor-non-anchor node pair $(i,l) \in \mathcal{E}_{as}$, the estimated distance between them is denoted by

$$d_{il}^2 = d_{il}^{\star 2} + \mu_{il},$$

where $\mu_{il} \in \mathbb{R}$ represents the measurement error. Here, $\mu_{il} \in \mathbb{R}$ captures anchor l's position uncertainty and is formulated in the form

$$\mu_{il} = \|\epsilon_l\|^2 - 2d_{il}^\star \|\epsilon_l\| \cos(\theta_{il}) + e_{il},$$

where $\theta_{il} \in [0, \pi]$ is the deviation angle from vector $x_i^\star - x_l^\star$ to vector ϵ_l, $e_{il} \in \mathbb{R}$ is a bias term. It is clear that if anchor positions are perfectly known, then $\epsilon_l = 0$ for $l \in \mathcal{N}_a$ and hence $\mu_{il} = e_{il}$.

On this basis, given the inaccurate locations x_l of anchor nodes and all noisy distance measurements d_{ij}, d_{il}, we aim to determine the locations of all non-anchor nodes and thereby formulate the noisy SNL problem:

$$\begin{aligned} \text{find} \quad & x_1, \ldots, x_N \in \mathbb{R}^2 \\ \text{s.t.} \quad & \|x_i - x_j\|^2 = d_{ij}^2, \forall (i,j) \in \mathcal{E}_{ss}, \\ & \|x_i - x_l\|^2 = d_{ij}^2, \forall (i,l) \in \mathcal{E}_{as}. \end{aligned} \tag{2}$$

Accordingly, denote $\boldsymbol{x}^\ddagger = \text{col}\{x_1^\ddagger, \ldots, x_N^\ddagger\} \in \mathbb{R}^{2N}$ as the solution to the noisy SNL problem (2).

A fundamental problem in noiseless and noisy SNL problems is the existence and uniqueness of solutions $\boldsymbol{x}^\dagger$ and $\boldsymbol{x}^\ddagger$ and their relationship with actual solution $\boldsymbol{x}^\star$. To this end, we formulate (2) as a multi-player potential game to reach $\boldsymbol{x}^\dagger$ and $\boldsymbol{x}^\ddagger$ from a game-theoretic perspective.

C. Potential game formulation

Here we formulate the noisy SNL problem as an N-player SNL potential game $G = \{\mathcal{N}_s, \{\Omega_i\}_{i\in\mathcal{N}_s}, \{J_i\}_{i\in\mathcal{N}_s}\}$, where $\mathcal{N}_s = \{1, \ldots, N\}$ corresponds to the player set, Ω_i is player i's local feasible set, which is convex and compact, and J_i is player i's payoff function. In this context, we map the position estimated by each non-anchor node as each player's strategy, i.e., the strategy of the player i (non-anchor node) is the estimated position $x_i \in \Omega_i$. Denote $\boldsymbol{\Omega} \triangleq \prod_{i=1}^N \Omega_i \subseteq \mathbb{R}^{2N}$, $\boldsymbol{x} \triangleq \text{col}\{x_1, \ldots, x_N\} \in \boldsymbol{\Omega}$ as the position estimate strategy profile for all players, and $\boldsymbol{x}_{-i} \triangleq \text{col}\{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N\} \subseteq \mathbb{R}^{n(N-1)}$ as the position estimate strategy profile for all players except player i. For $i \in \mathcal{N}_s$, the payoff function J_i is constructed as

$$J_i(x_i, \boldsymbol{x}_{-i}) = \sum_{j\in\mathcal{N}_s^i} (\|x_i - x_j\|^2 - d_{ij}^2)^2 + \sum_{l\in\mathcal{N}_a^i} (\|x_i - x_l\|^2 - d_{il}^2)^2,$$

where the first term in J_i measures the localization accuracy between non-anchor node i and its non-anchor node neighbor $j \in \mathcal{N}_s^i$ and the second term measures the localization accuracy between i and its anchor neighbor $l \in \mathcal{N}_a^i$.

The individual objective of each non-anchor node is to ensure its position accuracy, i.e.,

$$\min_{x_i\in\Omega_i} J_i\left(x_i, \boldsymbol{x}_{-i}\right). \tag{3}$$

Moreover, consider the following measurement of the overall performance of sensor nodes

$$\begin{aligned}
&\Phi(x_1, \ldots, x_N) \qquad (4)\\
&= \sum_{(i,j)\in\mathcal{E}_{ss}} (\|x_i - x_j\|^2 - d_{ij}^2)^2 + \sum_{(i,l)\in\mathcal{E}_{as}} (\|x_i - x_l\|^2 - d_{il}^2)^2\\
&= \sum_{(i,j)\in\mathcal{E}_{ss}} (\|x_i - x_j\|^2 - (d_{ij}^{\star 2} + \mu_{ij}))^2\\
&\quad + \sum_{(i,l)\in\mathcal{E}_{as}} (\|x_i - (x_l^\star + \epsilon_l)\|^2 - (d_{il}^{\star 2} + \|\epsilon_l\|^2 - 2d_{il}^\star \|\epsilon_l\| \cos(\theta_{il}) + e_{il}))^2.
\end{aligned}$$

Here, J_i denotes the localization accuracy of node i, which depends on the strategies of i's neighbors, while Φ denotes the localization accuracy of the entire network $\mathcal{G}$.

From a game-theoretic perspective, each sensing agent is considered as a selfish entity who simply tries to minimize its own payoff function. For this N-player game to provide a solution to localize the whole sensor network, each non-anchor node needs to consider the location accuracy of the whole sensor network while ensuring its own positioning accuracy through the given information. On this basis, we formulate $G = \{\mathcal{N}_s, \{\Omega_i\}_{i\in\mathcal{N}_s}, \{J_i\}_{i\in\mathcal{N}_s}\}$ as a potential game, where Φ in (4) satisfies the concept of a potential function [17], i.e.,

$$\Phi(x_i', \boldsymbol{x}_{-i}) - \Phi(x_i, \boldsymbol{x}_{-i}) = J_i(x_i', \boldsymbol{x}_{-i}) - J_i(x_i, \boldsymbol{x}_{-i}), \tag{5}$$

for every $i \in \mathcal{N}_s$, $\boldsymbol{x} \in \boldsymbol{\Omega}$, and unilateral deviation $x_i' \in \Omega_i$. This indicates that any unilateral deviation from a strategy profile always results in the same change in both individual payoffs and a unified potential function. In other words, the individual goal J_i is aligned with the global objective Φ.

Moreover, to attain an optimal value for $J_i(x_i, \boldsymbol{x}_{-i})$, players need to engage in negotiations and alter their optimal strategies. The best-known concept that describes an acceptable result achieved by all players is NE [10], [18].

Definition 1 (NE) *A profile $\boldsymbol{x}^\diamondsuit = \text{col}\{x_1^\diamondsuit, \ldots, x_N^\diamondsuit\} \in \boldsymbol{\Omega} \subseteq \mathbb{R}^{2N}$ is said to be an NE of game (3) if for any $i \in \mathcal{N}_s$,*

$$J_i\left(x_i^\diamondsuit, \boldsymbol{x}_{-i}^\diamondsuit\right) \leq J_i\left(x_i, \boldsymbol{x}_{-i}^\diamondsuit\right), \ \forall x_i \in \Omega_i. \tag{6}$$

We call NE as *global* NE due to the non-convex SNL formulation in this paper. This is different from the concept of *local* NE [19], which only satisfies condition (6) within a small neighborhood of $x_i^\diamondsuit$ for $i \in \mathcal{N}_s$, rather than whole Ω_i.

Definition 2 (local NE) *A strategy profile $\boldsymbol{x}^\diamondsuit$ is said to be a local NE of (3) if there exists a constant $r > 0$ such that for any $i \in \mathcal{N}_s$,*

$$J_i\left(x_i^\diamondsuit, \boldsymbol{x}_{-i}^\diamondsuit\right) \leq J_i\left(x_i, \boldsymbol{x}_{-i}^\diamondsuit\right), \ \forall x_i \in \Omega_i \cap \mathcal{B}_r(x_i^\diamondsuit). \tag{7}$$

where $\mathcal{B}_r(x_i^\diamondsuit) \triangleq \{y \in \mathbb{R}^2 : \|y - x_i^\diamondsuit\| < r\}$ is an open Euclidean ball with radius r and center $x_i^\diamondsuit$.

In order to reveal the relations between NE $\boldsymbol{x}^\diamondsuit$, noiseless node positions $\boldsymbol{x}^\dagger$ and noisy node positions $\boldsymbol{x}^\ddagger$, we need to utilize graph rigidity theory [15] to make the following generic and feasible assumption.

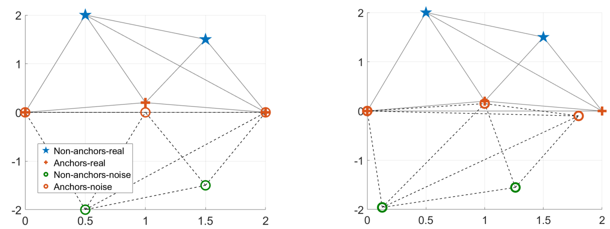
Assumption 1 *The sensor topology graph $\mathcal{G}$ is undirected and generically globally rigid.*

The generic global rigidity of $\mathcal{G}$ has been widely employed in SNL problems without measurement noises to make the geometric realization of the graph invariant, which indicates unique localization of the sensor network [15], [20].

D. Existence, uniqueness, and errors of the solution

As for the noiseless case, the global rigidity of the sensor network graph in Assumption 1 guarantees that (1) has a unique solution $\boldsymbol{x}^\dagger$ equal to the actual positions $\boldsymbol{x}^\star$ [20]. We uniformly use $\boldsymbol{x}^\star$ to represent $\boldsymbol{x}^\dagger$ hereafter.

However, in the noisy case, obtaining the relationship among $\boldsymbol{x}^\diamondsuit$, $\boldsymbol{x}^\ddagger$, and $\boldsymbol{x}^\star$ is not as straightforward. The presence of noise may perturb the rigidity of the graph, resulting in flip ambiguities or even no solution to the problem. Fig. 1(a) considers a case with inaccuracies in anchor node position information. The red pluses denote the true anchor node locations, while the blue stars represent the true non-anchor node locations. The grey lines indicate connections in this configuration. On the other hand, the red circles denote the noisy anchor node location information, while the green circles denote the computed noisy non-anchor node locations. The dashed lines indicate the noisy inter-node distance measurements. It can be seen that both anchor positions and inter-node distances among nodes in the actual configuration are close to those in the noisy one. However, the calculated or estimated positions of the non-anchor nodes become flipped compared to the actual positions. Furthermore, consider another case with errors in both anchor node position information and inter-node distance measurements. As shown in Fig. 1(b), even small errors may lead to the final localization results deviating largely from the actual positions.



(a) (b)

Fig. 1. A sensor network with two non-anchor nodes and three anchor nodes in two configurations.

Instead, consider a threshold for measurement noises such that if the errors in anchor node position and inter-node distance measurements do not exceed it, the extreme cases mentioned above would not occur and the node positions would remain close to the correct values. Therefore, the questions of interest to us are as follows:

(i) Under what conditions on noise levels does there exist a unique NE $\boldsymbol{x}^\diamondsuit$ and when does it correspond to the solution of the SNL problem (2)?

(ii) How do the errors between actual positions $\boldsymbol{x}^\star$ and NE $\boldsymbol{x}^\diamondsuit$ depend on the measurement noise amplitudes?

III. Existence and Uniqueness of NE

In this section, we investigate the existence and uniqueness of global NE $\boldsymbol{x}^\diamondsuit$. For the noiseless case, the following theorem reveals that the global NE $\boldsymbol{x}^\diamondsuit$ exists and is unique, corresponding to the actual non-anchor nodes' positions $\boldsymbol{x}^\star$.

Theorem 1 *Under Assumption 1, there exists a unique NE $\boldsymbol{x}^\diamondsuit$ of the game G, which satisfies $x_i^\diamondsuit = x_i^\star$ for $i \in \mathcal{N}_s$ and $x_l = x_l^\star$ for $l \in \mathcal{N}_a$ if $\mu_{ij} = 0$ for $(i,j) \in \mathcal{E}_{ss}$, $\mu_{il} = 0$ for $(i,j) \in \mathcal{E}_{as}$ and $\epsilon_l = 0$ for $l \in \mathcal{N}_a$.*

On the other hand, when the measurement noises arise, we investigate under what conditions of noise levels a unique local NE exists and is near the true sensor node locations. Denote $\boldsymbol{\mu} = \mathrm{col}\{\mu_{ij}\}_{(i,j)\in\mathcal{E}_{ss}}$, $\boldsymbol{e} = \mathrm{col}\{e_{il}\}_{(i,l)\in\mathcal{E}_{as}}$, $\boldsymbol{\epsilon} = \mathrm{col}\{\epsilon_l\}_{l\in\mathcal{N}_a}$, $\boldsymbol{\mu}_0 = \mathrm{col}\{\mu_{ij} = 0\}_{(i,j)\in\mathcal{E}_{ss}}$, $\boldsymbol{e}_0 = \mathrm{col}\{e_{il} = 0\}_{(i,l)\in\mathcal{E}_{as}}$, $\boldsymbol{\epsilon}_0 = \mathrm{col}\{\epsilon_l = 0\}_{l\in\mathcal{N}_a}$. The following lemma is established for the SNL problem with measurement noise.

Lemma 1

i) Let Assumption 1 hold. Then, there exists a small positive δ and positive constants a, b and c depending on δ such that for any constant uncertainty vectors $\boldsymbol{\mu}$, $\boldsymbol{e}$, and $\boldsymbol{\epsilon}$ satisfying $a\|\boldsymbol{\mu}\|^2 + b\|\boldsymbol{e}\|^2 + c\|\boldsymbol{\epsilon}\|^2 < \delta$, there is a unique solution $\hat{\boldsymbol{x}} = \mathrm{col}\{\hat{x}_1, \ldots, \hat{x}_N\} \in \boldsymbol{\Omega} \subseteq \mathbb{R}^{2N}$ of the equations $\nabla\Phi = 0$ satisfying $\|\hat{\boldsymbol{x}} - \boldsymbol{x}^\star\|^2 < \delta$.

ii) Let Assumption 1 hold. Then, for constant uncertainty vectors $\boldsymbol{\mu}$, $\boldsymbol{e}$ and $\boldsymbol{\epsilon}$, the solution $\hat{\boldsymbol{x}}$ is a local NE (with respect to $\boldsymbol{x}$) of $\Phi(\boldsymbol{x}, \boldsymbol{\mu}, \boldsymbol{e}, \boldsymbol{\epsilon})$.

Lemma 1 indicates that small magnitudes of measurement noises and uncertainty vectors will not disrupt the global rigidity of the graph, i.e., the positive definiteness of the Hessian matrix $\nabla^2\Phi$. Accordingly, the deviation of a local NE from the actual sensor positions will be bounded and continuously converge to zero as the errors in anchor node position information and inter-node distance measurements approach zero. Moreover, we can establish that when the measurement errors are constrained, the local NE $\hat{\boldsymbol{x}}$ is also the unique global NE $\boldsymbol{x}^\diamondsuit$, which corresponds to the unique solution $\boldsymbol{x}^\ddagger$ of noisy SNL problem (2). Moreover, the position errors between NE $\boldsymbol{x}^\diamondsuit$ and actual positions $\boldsymbol{x}^\star$ can be quantified, as detailed in the following theorem.

Theorem 2 *Let Assumption 1 hold. There exists a small positive δ and positive constants a, b and c depending on δ such that if $\boldsymbol{\mu}$, $\boldsymbol{e}$, and $\boldsymbol{\epsilon}$ satisfy $a\|\boldsymbol{\mu}\|^2 + b\|\boldsymbol{e}\|^2 + c\|\boldsymbol{\epsilon}\|^2 < \delta$, then*

i) there exists a unique NE $\boldsymbol{x}^\diamondsuit$, which is equal to $\boldsymbol{x}^\ddagger$;

ii) the NE $\boldsymbol{x}^\diamondsuit$ satisfies $\|\boldsymbol{x}^\diamondsuit - \boldsymbol{x}^\star\|^2 < \delta$.

Theorem 2 shows that when the measurement noises are not large, the NE is unique and returns sensor position estimates that are close to the actual positions. This establishes that a network can be approximately localized when the inter-node distance measurements and anchor node positions are contaminated with sufficiently small errors.

It is important to determine the value of δ and the parameters of a, b, and c for the NE. We first introduce the following notations.

Let $\mathcal{B}_{\delta_1}(\boldsymbol{x}^\star)$ denote the ball around $\boldsymbol{x}^\star$ defined by $\|\boldsymbol{x} - \boldsymbol{x}^\star\|^2 < \delta_1$, where δ_1 is a positive constant. Let $\mathcal{B}^c_{\delta_1}(\boldsymbol{x}^\star)$ denote the complementary set $\|\boldsymbol{x} - \boldsymbol{x}^\star\|^2 \geq \delta_1$. Observe that for all $\boldsymbol{\mu}$, $\boldsymbol{e}$ and $\boldsymbol{\epsilon}$ with $a\|\boldsymbol{\mu}\|^2 + b\|\boldsymbol{e}\|^2 + c\|\boldsymbol{\epsilon}\|^2 < \delta_1$, Lemma 1 guarantees that $\hat{\boldsymbol{x}} \in \mathcal{B}_{\delta_1}(\boldsymbol{x}^\star)$. Let Φ_1 be defined by

$$\Phi_1 = \min_{x_i, i\in\mathcal{N}_s, \boldsymbol{x}\in\mathcal{B}^c_{\delta_1}(\boldsymbol{x}^\star)} \sum_{(i,j)\in\mathcal{E}_{ss}} (\|x_i - x_j\|^2 - d_{ij}^{\star 2})^2 + \sum_{(i,l)\in\mathcal{E}_{as}} (\|x_i - x_l^\star\|^2 - d_{il}^{\star 2})^2. \tag{8}$$

On the other hand, consider also a collection of minimization problems, parameterized by a nonnegative constant δ_2, with variables $\boldsymbol{x}$, $\boldsymbol{\mu}$, $\boldsymbol{e}$ and $\boldsymbol{\epsilon}$:

$$\Phi_2 = \min_{x_i, i\in\mathcal{N}_s, \boldsymbol{x}\in\mathcal{B}^c_{\delta_1}(\boldsymbol{x}^\star), a\|\boldsymbol{\mu}\|^2 + b\|\boldsymbol{e}\|^2 + c\|\boldsymbol{\epsilon}\|^2 \leq \delta_2} \sum_{(i,j)\in\mathcal{E}_{ss}} (\|x_i - x_j\|^2 - d_{ij}^{\star 2} - \mu_{ij})^2 + \sum_{(i,l)\in\mathcal{E}_{as}} (\|x_i - (x_l^\star + \epsilon_l)\|^2 - (d_{il}^{\star 2} + \|\epsilon_l\|^2 - 2d_{il}^\star\|\epsilon_l\|\cos(\theta_{il}) + e_{il}))^2. \tag{9}$$

Then, we propose the following algorithm to determine the value of δ, where R denotes a large enough constant.

There are three reasons for limiting the value of δ.

1) To ensure the positive definiteness of $\nabla^2\Phi$. Note that

Algorithm 1

Initialization: error bounds δ_1, δ_2, constants a, b, c and R, uncertainty vectors $\boldsymbol{\mu}$, $\boldsymbol{e}$, and $\boldsymbol{\epsilon}$

1) *Input*: δ_1, a, b, and c, $\boldsymbol{\mu}$, $\boldsymbol{e}$ and $\boldsymbol{\epsilon}$ with $a\|\boldsymbol{\mu}\|^2+b\|\boldsymbol{e}\|^2+c\|\boldsymbol{\epsilon}\|^2 \leq \delta_1$

Solve: **while** $\nabla^2\Phi(\boldsymbol{x},\boldsymbol{\mu},\boldsymbol{e},\boldsymbol{\epsilon}) \preceq 0$ **do**

reset δ_1, $\boldsymbol{\mu}$, $\boldsymbol{e}$ and $\boldsymbol{\epsilon}$ such that $a\|\boldsymbol{\mu}\|^2+b\|\boldsymbol{e}\|^2+c\|\boldsymbol{\epsilon}\|^2 < \delta_1$

end while

solve (8) and obtain Φ_1 and δ_1

Output: $\frac{1}{2}\Phi_1$, δ_1

2) *Input*: δ_1, δ_2, a, b, c, R, $\frac{1}{2}\Phi_1$

Solve: **while** $\delta_2 > 0$ **do**

solve (9) and obtain Φ_2

if $\Phi_2 \geq \frac{1}{2}\Phi_1$ for all $\delta_2 \leq \delta_1$

$\delta_2 := \delta_1$

break

else

choose δ_2 such that $\Phi_2 = \frac{1}{2}\Phi_1$

break

end if

end while

Output: δ_2

3) *Input*: error bound δ_2, $\frac{1}{2}\Phi_1$

Solve: $\delta := \min\{\delta_2, \frac{1}{2}\Phi_1\}$

Output: δ

$\nabla^2\Phi$ is positive definite at $(\boldsymbol{\mu}_0, \boldsymbol{e}_0, \boldsymbol{\epsilon}_0)$, but may cease to hold well away from this point. Thus, we need to limit the size of δ_1 in Step 1 of Algorithm 1.

2) To avoid getting stuck into local NE in noisy sensor localization. In the noiseless case, there may exist a local NE (denoted as $\hat{\boldsymbol{x}}$), whose value $\Phi(\hat{\boldsymbol{x}})$ differs from zero by a small amount. The coordinate values corresponding to local NE $\hat{\boldsymbol{x}}$ are different from those corresponding to the global NE $\boldsymbol{x}^\diamondsuit = \boldsymbol{x}^\star$. Then, when the noise levels are slowly increased from zero, the coordinate values corresponding to the global NE $\boldsymbol{x}^\diamondsuit$ may jump at some noise level to the coordinate values corresponding to local NE $\hat{\boldsymbol{x}}$. To avoid this situation, we should take $a\|\boldsymbol{\mu}\|^2 + b\|\boldsymbol{e}\|^2 + c\|\boldsymbol{\epsilon}\|^2 \leq \delta_2 < \delta_1$ to ensure $\Phi(\hat{\boldsymbol{x}})$ is not smaller than $\frac{1}{2}\Phi_1$. Therefore, $\Phi(\hat{\boldsymbol{x}})$ should be larger enough than $\Phi(\boldsymbol{x}^\diamondsuit)$.

3) To ensure that the global NE $\boldsymbol{x}^\diamondsuit$ in noisy sensor localization is close to the real localization $\boldsymbol{x}^\star$. To achieve this, we should take $a\|\boldsymbol{\mu}\|^2+b\|\boldsymbol{e}\|^2+c\|\boldsymbol{\epsilon}\|^2 < \frac{1}{2}\Phi_1$, so as to let $\Phi(\boldsymbol{x}^\diamondsuit)$ not greater than $\frac{1}{2}\Phi_1$. Hence, $\boldsymbol{x}^\diamondsuit$ is not far from the actual positions $\boldsymbol{x}^\star$.

To determine the values of a, b and c, we can follow two steps. Firstly, we can employ the implicit function theorem [21] or Lipschitz continuity to obtain a, b, and c based on Lemma 1. Secondly, we compare them respectively with $a' = 1$, $b' = 2$ and $c' = 32R_s^2$. If $a < 1$ or $b < 2$ or $c < 32R_s^2$, then we set $a = 1$, $b = 2$ and $c = 32R_s^2$; otherwise, we keep them unchanged. The reason for limiting a, b, and c is not only to satisfy $\|\boldsymbol{x}^\diamondsuit - \boldsymbol{x}^\star\| < \delta$ but also to ensure $\Phi(\boldsymbol{x}^\diamondsuit) < \frac{1}{2}\Phi_1$.

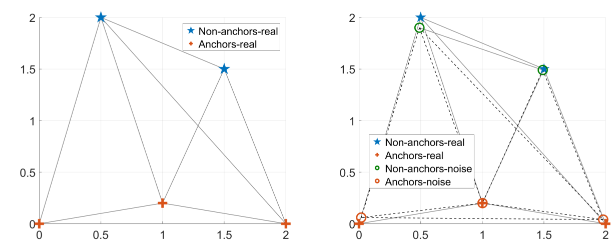
Finally, we apply our results in two case studies. In the case that the anchor's positions are perfectly known, i.e., $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}_0 = \mathrm{col}\{0\}_{l=1}^{|\mathcal{N}_a|}$, the results in Theorem 2 can be simplified.

Corollary 1 *Under Assumption 1, there exists a small positive δ and positive constants $\bar{a}$ depending on δ such that if $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}_0$ and $\boldsymbol{\mu}$ and $\boldsymbol{e}$ satisfy $\|\boldsymbol{\mu}\| + \|\boldsymbol{e}\| < \delta$, then the NE of SNL problem* (3) *is unique and $\|\boldsymbol{x}^\diamondsuit - \boldsymbol{x}^\star\| \leq \bar{a}(\|\boldsymbol{\mu}\| + \|\boldsymbol{e}\|)$.*

In the case that the inter-node distance measurement errors are zero, i.e., $\boldsymbol{\mu} = \boldsymbol{\mu}_0 = \mathrm{col}\{\mu_{ij} = 0\}_{(i,j)\in\mathcal{E}_{ss}}$ and $\boldsymbol{e} = \boldsymbol{e}_0 = \mathrm{col}\{e_{il} = 0\}_{(i,l)\in\mathcal{E}_{as}}$, we have the following results.

Corollary 2 *Under Assumption 1, there exists a small positive δ and positive constants $\bar{b}$ depending on δ such that if $\boldsymbol{\mu} = \boldsymbol{\mu}_0$, $\boldsymbol{e} = \boldsymbol{e}_0$ and $\boldsymbol{\epsilon}$ satisfies $\|\boldsymbol{\epsilon}\| < \delta$, then the NE of SNL problem* (3) *is unique and $\|\boldsymbol{x}^\diamondsuit - \boldsymbol{x}^\star\| \leq \bar{b}\|\boldsymbol{\epsilon}\|$.*

IV. Numerical Experiments



(a) True localization (b) Noisy localization

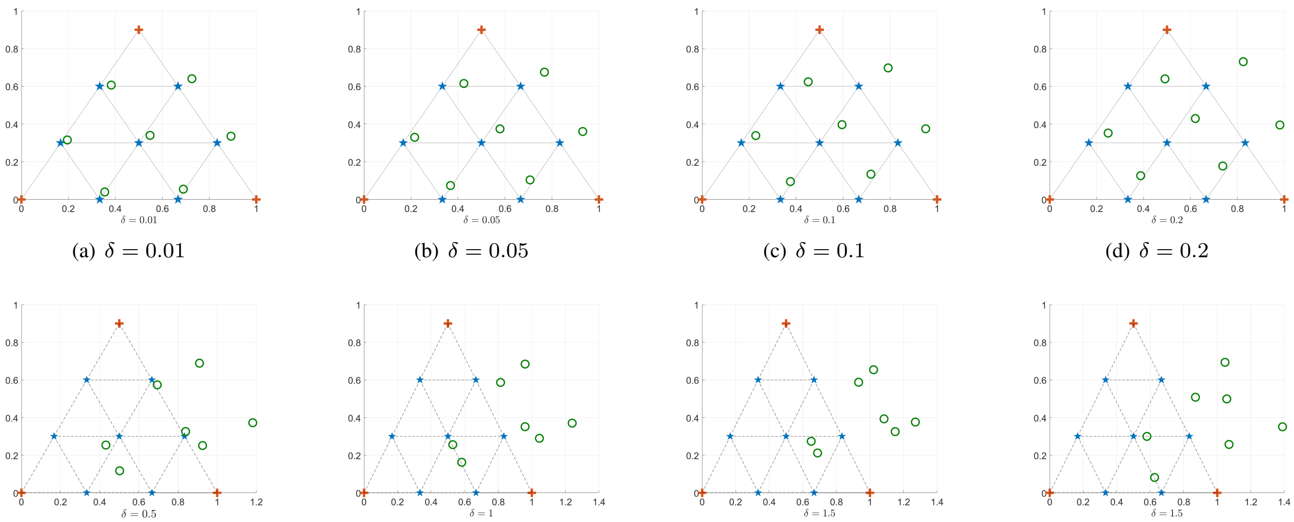
Fig. 2. The true localization and noisy localization results.

Consider an SNL problem with noisy measurements. Take $N = 2$ non-anchor nodes and $M = 3$ anchor nodes. The original configuration is shown in Fig. 2(a). The true positions of anchor and non-anchor nodes are represented by blue stars and red pluses respectively, consistent with Fig. 1. The gray lines indicate connections between nodes. Following the procedures in Alg.1, we set $\delta_1 = 0.1$, $a = 1$, $b = 32$, $c = 2$ and get $\delta = 0.1$. The corresponding noisy localization result is shown in Fig. 2(b). We can see from Fig. 2(b) that when the measurement errors satisfy the condition in Theorem 2, the position errors between computed results and actual ones are also small.

Next, we consider another configuration with $N = 7$ non-anchor nodes and $M = 3$ anchor nodes [8]. Fig. 3 shows the noisy localization results under different error bounds. As δ ranges from 0.01 to 0.2, the noisy localization result (shown as green circles) gradually deviates from the true values (shown as blue stars), while its graph structure remains globally rigid, similar to the true one. However, when δ exceeds 0.5, the graph structure gradually becomes distorted. Particularly, at $\delta = 2$, such measurement errors completely disrupt the final computation results.

V. Concluding Remarks

In this paper, we have studied the solution of the SNL problems with noisy measurements. By formulating a non-convex SNL potential game, we have investigated the existence and uniqueness of NE in both ideal settings with



(a) $\delta = 0.01$ (b) $\delta = 0.05$ (c) $\delta = 0.1$ (d) $\delta = 0.2$

(e) $\delta = 0.5$ (f) $\delta = 1$ (g) $\delta = 1.5$ (h) $\delta = 2$

Fig. 3. Computed sensor location results with different error levels.

accurate anchor node location information and accurate inter-node distance measurements and practical settings with anchor location inaccuracies and distance measurement noise. In the noiseless case, we have shown that the NE exists and is unique, and corresponds to the precise network localization. In the case involving errors in anchor node position and inter-node distance measurements, we have established that if these errors are sufficiently small, the NE exists and is still unique, providing an approximate solution to the SNL problem. Moreover, the position errors from NE to the precise network localization can be quantified, providing that the measurement errors are constrained by a small bound.

In follow-up works, we plan to extend our results to more complex scenarios including i) designing an NE-seeking algorithm with global convergence, ii) generalizing the model to a distributed case, and iii) exploring milder graph conditions.

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